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Truss Optimization on Shape and Sizing with Frequency Constraints

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An optimality criteria algorithm is presented for three-dimentional truss structure optimization with multiple constraints on its natural frequencies. Both nodal coordinates and element cross-sectional areas, which are quite different in their natures, are treated simultaneously in a unified design space for structural weight minimization. First the optimality criterion is developed for a single constraint based on differentiation of the Lagrangian function. It states that, at the optimum, all of the variables should have equal efficiencies. Then, the global sensitivity numbers are introduced to solve multiple constraints of frequencies, avoiding computation of the Lagrange multipliers. Finally, upon the sensitivity analysis, the most efficient variables are identified and modified in priority. The optimal solution is achieved gradually from the initial design with a minimum weight increment. Four typical trusses are used to demonstrate the feasibility and validity of the proposed method.

I. Introduction

THE optimal design of a truss structure with dynamic constraints has been an active research topic for many years. Important progress has been made in both optimality criteria and solution techniques.[1–8] However, due to the highly nonlinear characteristics of dynamic optimization problems, the bulk of the literature has been devoted mainly to truss sizing optimization where the structural shape (geometry) is generally prescribed according to the designer's experience.[4–7] Moreover, the structure is often attached with nonstructural masses to scale the sizing variables effectively to obtain an initially feasible design.[4,5] As is well known, the optimal design of a truss shape depends not only on its topology, but also on the distribution of element cross-sectional areas. On the other hand, the structural shape has a great impact on its sizing optimization. This inherent coupling of structural shape and element sections explicitly indicates that truss shape or sizing optimization should not be performed independently. However, shape and sizing variables are fundamentally different physical representations. Combining these two types of variables may entail considerable mathematical difficulties and, sometimes, lead to ill-conditioning problems because their changes are of widely different orders of magnitude.

To date, relatively little literature is available on truss shape and sizing simultaneous optimization with dynamic constraints. Lin et al.[1] developed an optimality criteria algorithm based on two damping factors for problems with frequency-prohibited band constraints. Sadek[2] presented a two-step algorithm consisting of frequency correction and weight reduction schemes with a fundamental frequency constraint. The method was based on the Kuhn–Tucker (K–T) optimality conditions for structural weight minimization. Sergeyev and Mroz[8] applied a sequential quadratic programming (SQP) technique for optimal design of a frame structure with both shape and sizing variables.

The purpose of the present study is to extend the evolutionary node shift method, basically developed for truss shape optimization,[9,10] to more general situations of truss shape and sizing optimization with multiple constraints of natural frequencies. The most challenging part of this task is to optimize simultaneously two different types of variables, node coordinates and element cross-sectional areas of a truss structure, in one design space without any separation. These variables will be optimally redesigned in a unified approach for structural weight minimization. By differentiation of the Lagrangian function with respect to the variables, the optimality criterion is found for all of the variables, regardless of their properties. This criterion states that, at the optimum, all of the design variables should have equal sensitivity numbers or efficiencies. By means of the sensitivity analysis, the most efficient variables can be identified and optimized systematically. To avoid evaluating the Lagrange multipliers, the weighted sum of the sensitivity numbers is used to resolve multiple constraints of frequencies. In each iterative loop, sensitivity analysis helps to find the search direction and to determine the change interval of a design variable accordingly. The optimal solution is achieved gradually from an infeasible starting point with a minimum weight increment, and the structural weight is indirectly minimized. The proposed method is examined with four typical truss structures and is shown to be quite effective and reliable.

II. Problem Formulation

In truss shape and sizing optimization problems, the structural topology is prescribed in advance and kept fixed in the solution process. Nodal coordinates and element cross-sectional areas are referred to as the design variables and are assumed to change continuously in this context. The natural frequencies are posed as constraints for the structure to avoid resonance with the external excitations. In addition, each variable may be constrained within an acceptable region. Thus, the optimization problem can be defined mathematically as follows:

Minimize

$$W = \sum_{e=1}^{n} L_e \rho_e A_e \tag{1}$$

subject to

$$\omega_i \geq \omega_i^*, \qquad i = 1, \ldots, q_1 \tag{2}$$

$$\omega_i \leq \omega_i^*, \qquad i = q_1 + 1, \ldots, q \tag{3}$$

$$\underline{v}_j \leq v_j \leq \bar{v}_j, \qquad j = 1, \ldots, k \tag{4}$$

where W is the structural weight. L_e, ρ_e, and A_e are the length, material density, and cross-sectional area of the eth element, respectively. Here n is the total number of elements in the structure and k the number of the independent design variables. Inequality (2) represents that some natural frequencies ω_i, numbering q_1, should

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exceed the prespecified lower limits. In inequality (3), other natural frequencies ω_i, numbering q–q_1, must be less than the corresponding upper limits, respectively. Inequality (4) indicates that the design variable v_j, including either a shape or a sizing variable, should take a value between its lower bound $\underline{v}_j$ and upper bound $\bar{v}_j$, respectively.

In engineering practice, a number of shape or sizing variables are linked to keep the structure symmetric and/or limit the number of variables treated as

$$v_d = f(v_j) \tag{5}$$

where v_d is a dependent variable and v_j is an independent one.

Note that the influences of shape and sizing variables on both the objective and constraints are fairly different. Moreover, the natural frequency is highly nonlinear and implicit with respect to the design variables.

III. Optimality Criterion

At the optimum, the solution should satisfy the K–T optimality conditions (see Ref. 11). For a single frequency constraint with a lower bound, the Lagrangian function of a weight minimization problem is formulated as

$$L(\boldsymbol{V}) = W(\boldsymbol{V}) - \lambda\left[\omega_i^2(\boldsymbol{V}) - \left(\omega_i^*\right)^2\right] \tag{6}$$

In which $\boldsymbol{V}$ is a vector representing all of the design variables and λ is the Lagrange multiplier. When the Lagrangian function is differentiated with regard to variables, the K–T optimality conditions are constructed as[11]

$$\begin{gathered} \nabla_V L(\boldsymbol{V}) = \nabla_V W(\boldsymbol{V}) - \lambda \nabla_V \omega_i^2(\boldsymbol{V}) = \{0\} \\ \omega_i^2(\boldsymbol{V}) - \left(\omega_i^*\right)^2 \geq 0 \\ \lambda\left[\omega_i^2(\boldsymbol{V}) - \left(\omega_i^*\right)^2\right] = 0 \\ \lambda \geq 0 \end{gathered} \tag{7}$$

In most constrained optimization problems, it is well known that the optimum point lies on the constraint boundary separating the feasible and infeasible regions. In such a case, $\partial W/\partial v_j \neq 0$. Then from the first expression of Eq. (7), which actually contains k separate equations, one obtains

$$\frac{1}{\lambda} = \frac{\partial \omega_i^2}{\partial v_j} \Bigg/ \frac{\partial W}{\partial v_j} = \frac{\partial \omega_i^2}{\partial W} = \alpha_{ij}, \qquad j = 1, \ldots, k \tag{8}$$

where α_{ij} is defined, in this paper, as the sensitivity number of each design variable, which indicates the ratio of the effect on the ith natural frequency to the effect on the structural weight due to the change of the jth design variable.[10] It is understandable from Eqs. (7) and (8) that, at the optimum, all of the variables ought to have the same positive sensitivity number, no matter what the nature of the variable is. On the other hand, it is obvious from the definition that α_{ij} also denotes the efficiency of the modification of the jth design variable. The largest value of α_{ij} implies the larger increment of ω_i together with the smaller increment of structural weight. Therefore, Eq. (8) also implies, by intuition, that all of the design variables should have equal efficiencies at the optimum.

Nonetheless, it may be rather cumbersome to determine the optimal solution based on the equality of all sensitivity numbers α_{ij} if there exist a large number of design variables in a practical problem. An alternative optimality condition is suggested according to the gradient of the Lagrangian function.

Premultiplying both sides of the first expression of Eq. (7) by $(\nabla\omega_i^2)^T$ yields

$$\lambda = \left(\nabla\omega_i^2\right)^T \nabla W \Big/ \left(\nabla\omega_i^2\right)^T \left(\nabla\omega_i^2\right) \tag{9a}$$

By the substitution of Eq. (9a) into the first expression of Eq. (7), if the gradient of the Lagrangian function satisfies the prescribed termination condition

$$\|\nabla_V L(\boldsymbol{V})\| \leq 10^{-2} \tag{9b}$$

then the design point is believed to be convergent, and the optimization process may be terminated.

Note that in the preceding derivations, we have assumed that all of the design variables can be changed freely, that is, the design variable does not arrive at its lower or upper bound. However, side constraints are more often than not involved in the optimization process, especially for sizing variables. For instance, the minimum sectional area may typically represent requirements on the strength, buckling, or fabrication of the member. If a design variable can be changed forward or backward, it is referred to as an active variable. On the other hand, once a variable reaches its lower or upper bound, it could not be modified in the associated direction further and is regarded as a passive variable in the corresponding direction. Passive variables should be eliminated in both the optimality criterion of Eq. (8) and the convergent condition of Eq. (9).

IV. Calculation of Sensitivity Numbers

A. Single Frequency Constraint

In dynamic analysis, the generalized eigenvalue problem of an undamped structure is represented by

$$\left([K] - \omega_i^2[M]\right)\{\phi\}_i = \{0\} \tag{10}$$

where $[K]$ and $[M]$ are the global stiffness and mass matrices, respectively, ω_i is the ith natural frequency, and $\{\phi\}_i$ is the related vibration mode of the structure, which has been normalized such that

$$\{\phi\}_i^T [M]\{\phi\}_i = 1 \tag{11}$$

Generally, the derivatives of the global stiffness matrix $[K]$ and mass matrix $[M]$ with respect to the jth design variable v_j can be evaluated at element level[11]:

$$\frac{\partial[K]}{\partial v_j} = \sum_{e=1}^{n_j} \frac{\partial[k_e]}{\partial v_j}, \qquad \frac{\partial[M]}{\partial v_j} = \sum_{e=1}^{n_j} \frac{\partial[m_e]}{\partial v_j} \tag{12}$$

where $[k_e]$ and $[m_e]$ are the element stiffness and mass matrices, respectively. Here n_j is the number of elements associated with the jth design variable. As a result, the derivative of a distinct frequency with respect to the jth design variable is given as[11]

$$\begin{aligned} \frac{\partial\omega_i^2}{\partial v_j} &= \{\phi\}_i^T \left(\frac{\partial[K]}{\partial v_j} - \omega_i^2 \frac{\partial[M]}{\partial v_j}\right)\{\phi\}_i \\ &= \sum_{e=1}^{n_j} \{\phi_e\}_i^T \left(\frac{\partial[k_e]}{\partial v_j} - \omega_i^2 \frac{\partial[m_e]}{\partial v_j}\right)\{\phi_e\}_i, \qquad j = 1, \ldots, k \end{aligned} \tag{13}$$

In the preceding expression, $\{\phi_e\}_i$ is the ith mode of the eth element, which contains the related entries of $\{\phi\}_i$. The derivative of the structural weight is

$$\frac{\partial W}{\partial v_j} = \sum_{e=1}^{n_j} \rho_e \frac{\partial(A_e L_e)}{\partial v_j} \tag{14}$$

Thus, the sensitivity number of the jth design variable is computed as

$$\alpha_{ij} = \frac{\partial\omega_i^2}{\partial v_j} \Bigg/ \frac{\partial W}{\partial v_j} = \sum_{e=1}^{n_j} \{\phi_e\}_i^T \left(\frac{\partial[k_e]}{\partial v_j} - \omega_i^2 \frac{\partial[m_e]}{\partial v_j}\right)\{\phi_e\}_i \Bigg/ \left[\sum_{e=1}^{n_j} \rho_e \frac{\partial(A_e L_e)}{\partial v_j}\right], \qquad j = 1, \ldots, k \tag{15}$$

The frequency increment can be represented linearly in terms of each variable perturbation:

$$\Delta\omega_i^2 \approx \frac{\partial\omega_i^2}{\partial v_j} \cdot \Delta v_j, \qquad j = 1, \ldots, k \tag{16}$$

To increase a specified natural frequency ω_i, one could obtain the following relationship according to Eq. (16):

$$\mathrm{sign}(\Delta v_j) = \mathrm{sign}\left(\frac{\partial \omega_i^2}{\partial v_j}\right), \qquad j = 1, \ldots, k \tag{17}$$

where Δv_j is the change interval of the jth variable and sign(·) is the sign function. As a consequence, Eq. (17) can be used to determine the search direction of the jth design variable. Conversely, decreasing a specified natural frequency would yield the opposite sign expression.

For the case of increasing a specified frequency, the positive value of α_{ij} implies that ω_i and W increase simultaneously, whereas the negative value of α_{ij} suggests ω_i increasing along with W decreasing. In contrast, decreasing a specified frequency would lead to the opposite conclusions. Therefore, for minimum weight design, the sensitivity number calculated for each design variable can be utilized to identify the most efficient one and to find the associated search direction.

B. Multiple Constraints of Frequencies

In practical designs, several frequencies of interest may be constrained as in Eqs. (2) and (3). Sometimes, a frequency-prohibited band requirement may be imposed on the structure to control the dynamic response in forced excitation problems.[1] To resolve multiple constraints of natural frequencies, the Lagrange multipliers in the K–T optimality conditions should be determined for all active constraints. Grandhi and Venkayya[4] had to solve nonlinear sets of equations, whereas Khot[5] developed a set of simultaneous linear equations. However, this paper adopts the approach of weighted sum of the sensitivity numbers to resolve several frequency constraints[10,12]:

$$\eta_j = \sum_{i=1}^{q} \overline{\lambda_i} \alpha_{ij}, \qquad j = 1, \ldots, k \tag{18}$$

where η_j is defined as the global sensitivity number with respect to the jth design variable in the case of multiple constraints. Here $\overline{\lambda_i}$ is the weighting coefficient of each frequency constraint, which depends mainly on the degree of violation for the related constraint. Note that $\overline{\lambda_i}$ is not the corresponding Lagrange multiplier and is proposed herein as

$$\overline{\lambda_i} = \left(\omega_i^* / \omega_i\right)^b, \qquad i = 1, \ldots, q_1 \tag{19}$$

for the case of increasing a frequency, or

$$\overline{\lambda_i} = -\left(\omega_i / \omega_i^*\right)^b, \qquad i = q_1 + 1, \ldots, q \tag{20}$$

for the case of decreasing a frequency.

The advantage of introducing the global sensitivity number η_j lies in that it provides a unified approach to deal with optimization problems of both single and multiple frequency constraints, as well as the requirements of increasing or decreasing a frequency. Moreover, the maximum value of η_j always corresponds to the most efficient design variable. The exponent b is a penalty factor. A typical value of $b = 5$ is used due to the consideration of the inherent coupling of vibration frequencies in the optimization process.

V. Sensitivity Number for Different Design Variables

In the optimization problem, the two types of design variables, nodal coordinates and element cross sections, are quite different in their units, as are the formulas for sensitivity number computation for each type of the variable. However, based on the design sensitivity number or the variable's efficiency, all design variables can be compared and then optimized in the similar manner, irrespective of their physical discrepancies.

A. Sensitivity Number of a Nodal Coordinate

The algorithm for computing the sensitivity number of a nodal coordinate has been described in detail in Ref. 10. Herein, the results are employed directly. The sensitivity number of the ith natural frequency with respect to the jth node shift along a particular axis in the Cartesian system is computed as follows:

$$\alpha_{ij} = \sum_{e=1}^{n_j} \{\phi_e\}_i^T \left(\frac{\partial [k_e]}{\partial x_j} - \omega_i^2 \frac{\partial [m_e]}{\partial x_j}\right) \{\phi_e\}_i \Bigg/ \left[-\sum_{e=1}^{n_j} \rho_e A_e \beta_e\right]$$

$$j = 1, \ldots, k_n \tag{21}$$

where β_e is the directional cosine of the eth element with respect to the related axis and k_n is the number of the possibly shifted nodes.

In the case of more than one frequency constraint, the shift interval of each node is determined by[10]

$$|\Delta x_j| = \frac{0.01}{q} \cdot \left(\sum_{i=1}^{q_1} \frac{\omega_i^*}{\omega_i} + \sum_{i=q_1+1}^{q} \frac{\omega_i}{\omega_i^*}\right) \cdot \sqrt{\frac{\alpha_{ij}}{\alpha_{i\,\min}}}$$

$$\cdot \,\mathrm{minimum}\{L_e, e = 1, n_j\} \tag{22}$$

where $\alpha_{i\,\min}$ is the minimum positive value of the sensitivity number, which corresponds to the least efficient variable.

B. Sensitivity Number of an Element Sectional Area

As is known, the element stiffness and mass matrices are linear functions of the element's sectional area, respectively. The derivatives of the global stiffness matrix $[K]$, mass matrix $[M]$, and structural weight W with respect to the jth cross-sectional area A_j are evaluated as

$$\frac{\partial [K]}{\partial A_j} = \sum_{e=1}^{n_j} \frac{\partial [k_e]}{\partial A_j} = \frac{1}{A_j} \sum_{e=1}^{n_j} [k_e] \tag{23a}$$

$$\frac{\partial [M]}{\partial A_j} = \sum_{e=1}^{n_j} \frac{\partial [m_e]}{\partial A_j} = \frac{1}{A_j} \sum_{e=1}^{n_j} [m_e] \tag{23b}$$

$$\frac{\partial W}{\partial A_j} = \sum_{e=1}^{n_j} \rho_e L_e \tag{24}$$

Therefore, the sensitivity number of the ith natural frequency to the jth sectional modification is computed as

$$\alpha_{ij} = \sum_{e=1}^{n_j} \{\phi_e\}_i^T \left([k_e] - \omega_i^2 [m_e]\right) \{\phi_e\}_i \Bigg/ \left[A_j \sum_{e=1}^{n_j} \rho_e L_e\right]$$

$$j = 1, \ldots, k_s \tag{25}$$

where k_s refers to as the number of the independent sizing variables. Analogous to the shape variables, the interval of each sizing modification is evaluated as

$$|\Delta x_j| = \frac{0.01}{q} \cdot \left(\sum_{i=1}^{q_1} \frac{\omega_i^*}{\omega_i} + \sum_{i=q_1+1}^{q} \frac{\omega_i}{\omega_i^*}\right) \cdot \sqrt{\frac{\alpha_{ij}}{\alpha_{i\,\min}}} \cdot A_j \tag{26}$$

Note that, although the shape and sizing variables are different in their units and orders of magnitudes, the units of their sensitivity numbers are the same. Thus the comparison and identification of their efficiencies make sense in a unified measurement.

VI. Evolutionary Optimization Procedure

To implement the optimization procedure with frequency constraints, the finite element method is used as an analyzer to calculate the natural frequencies and vibration modes of interest. With these results, the sensitivity numbers for the two kinds of the variables can be calculated simply. Generally, the computational cost for sensitivity numbers is trivial in contrast to that for solving structural eigenpairs.

A. Number of Variables Changed in Each Loop

In the proposed evolutionary optimization method, the number of the variables modified in each design loop is very important. If only one variable with the highest efficiency is changed, the process may be prohibitively time consuming. At times, the eigenpair calculation has to be carried out over thousands of times for complex structures. To reduce the computational expenses of modal analysis and sensitivity number calculation, more than one design variable with the larger sensitivity number may be changed in each iterative loop. Those that satisfy the condition

$$\{v_j | \alpha_{ij} \geq [\alpha_{i\,\max} - 20\%(\alpha_{i\,\max} - \alpha_{i\,\min})], \qquad j = 1, k\} \tag{27}$$

are changed in each loop, where $\alpha_{i\,\max}$ is the maximum sensitivity number (largest positive value).

According to the sensitivity analysis of design variables presented, the strategy to implement the truss optimization procedure is[10] changing preferentially all of the variables with negative sensitivity numbers and, then, the variables with the larger positive sensitivity numbers.

B. Repeated Frequencies

In the preceding presentation, we seldom refer to repeated natural frequencies. As is well known, repeated frequencies are typically associated with structural symmetry or are induced by the evolution of frequencies during the solution process. Usually, repeated frequencies are not differentiable in common sense. That is, the Frechet derivative does not exist due to the nonuniqueness of the associated modes; only directional derivatives can be found. However, the design sensitivities of a repeated frequency can be evaluated by solving a subeigenvalue problem,[13]

$$\left[[\Phi]^T\left(\frac{\partial[K]}{\partial v_j} - \omega_i^2\frac{\partial[M]}{\partial v_j}\right)[\Phi] - \frac{\partial\omega_i^2}{\partial v_j}I\right]\{\theta\} = 0 \tag{28}$$

where $[\Phi]$ consists of normalized modes corresponding to the repeated frequency, $\{\theta\}$ is a coefficient vector, and I is an identity matrix. In fact, the derivatives of the repeated frequency are irrelevant to the selection of the associated modes. When the derivatives of the repeated frequencies are arranged in ascending order, the sensitivity numbers of each variable can be computed as usual.

Generally, when two frequencies are closely spaced as

$$\omega_{j+1} - \omega_j \leq 10^{-3} \cdot \omega_j \tag{29}$$

they are thought to be coalescent and become a repeated frequency. Note that the coalescence of frequencies commonly results in mode switching and makes the optimality algorithm more complicated.[3]

VII. Numerical Examples

To demonstrate its feasibility and validity, the proposed method is applied to truss optimization with shape and sizing variables. The program is developed based on a general-purpose analyzer of SAMCEF®/Dynam commercial software. A consistent mass matrix is adopted in both the modal analysis and the sensitivity computation.

A. Two-Bar Truss

This simple example aims at comparing the efficiencies of shape and sizing variables. In Fig. 1, a symmetric two-bar truss structure with a nonstructural mass attachment $m = 10$ kg at the joint is designed for its shape and size optimization. The structure is subject to a single frequency constraint in two different cases. The fundamental frequency of the system is required to be greater than 60 and 100 Hz, respectively. One shape variable of the support position y_s and one sizing variable of the element cross-sectional area A can be redesigned in the process. Young's modulus is $E = 2.1 \times 10^{11}$Pa, and the material density is $\rho = 7800$ kg/m^3. Suppose the lower limit of the section to be 0.2 cm^2. Table 1 gives the initial and optimal designs, as well as sensitivity numbers for the two variables, respectively. Table 2 lists the natural frequencies and convergent conditions obtained. The evolutionary histories of the fundamental frequency with different constraint bounds are shown in Fig. 2.

It is found from Table 1 that the fundamental frequency increases to 60 Hz only by shape modification. The cross-sectional area remains fixed during the optimization. This is because the sensitivity number of the shape variable is much higher than that of the sizing variable in the whole process, as shown in Fig. 3. In the second case of 100-Hz limit, the supports are first shifted until the two frequencies coalesce and become a double repeated frequency (86.1 Hz) at $y_s = 1.00$ m. Afterward, the shape variable is no longer capable

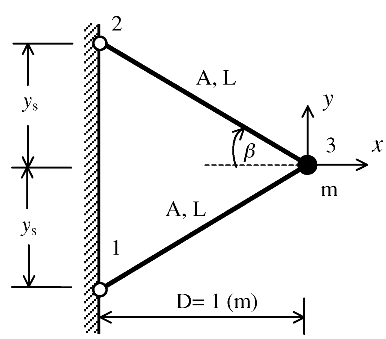

Fig. 1 Symmetric two-bar truss structure.

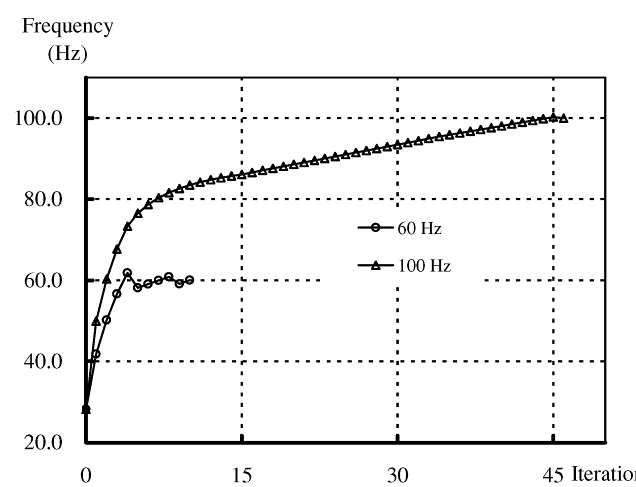

Fig. 2 Evolutionary histories of the fundamental frequency under different constraint bounds.

Table 1 Initial and optimum design variables under different constraint bounds

			$\omega_1 \geq 60$ Hz		$\omega_1 \geq 100$ Hz	
Design variable	Initial design	Sensitivity number	Optimal design	Sensitivity number	Optimal design	Sensitivity number[a]
y_s, m	0.2	4.827×10^6	0.485	3.073×10^6	1.000	-1.993×10^6, 6.475×10^5
A, cm^2	0.2	9.749×10^4	0.200	4.058×10^5	0.271	6.470×10^5, 6.473×10^5
Weight, kg		0.318		0.347		0.598

[a]Sensitivity numbers of a double repeated frequency.

Table 2 Natural frequencies for initial and optimal designs

Frequency order	Initial design	Optimal design $\omega_1 \geq 60$ Hz	Optimal design $\omega_1 \geq 100$ Hz
1	28.2	60.0	100.0
2	140.9	123.8	100.0
$\|\nabla L(X)\|$		2.58×10^{-5}	0.00

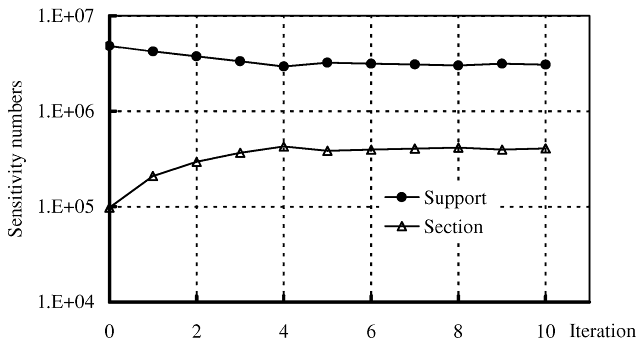

Fig. 3 Comparison of sensitivity numbers of shape and sizing variables.

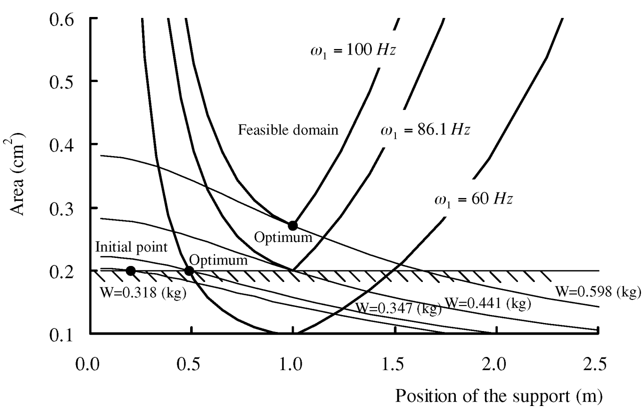

Fig. 4 Diagram of the optimal designs.

of increasing the fundamental frequency because the relevant sensitivity numbers of the shape variable are opposite in their signs, as can be seen in the last column in Table 1. Any more support shifting would give rise to switching of vibration modes and declining of the fundamental frequency. Subsequently, the sectional area of the elements is resized to raise the repeated fundamental frequency to 100 Hz steadily because its corresponding sensitivity numbers are the same, which means that it can raise the repeated frequencies identically in this situation. Figure 4 shows the optimization process diagrammatically. Because the sectional area of the element is still at its lower bound, there exists a great discrepancy in the sensitivity numbers in the first case, with the 60-Hz limit.

This schematic example shows that, generally, the shape variable is more efficient than the sizing variable. Just as pointed out by Sergeyev and Mroz,[8] the structure response is much more sensitive to node positions. Additionally, this example also reveals that shape optimization is limited to the rise of the natural frequency of a structure. The Appendix will give the theoretical analysis of this example.

B. Ten-Bar Truss in Sizing Optimization

This example has been investigated in more detail by Grandhi and Venkayya[4] using optimality criteria algorithm, as well as by Sedaghati et al.[7] using the SQP technique. Here, it is used to compare the results by the present method with their results.[4,7]

The structure, shown in Fig. 5, is made of aluminum with Young's modulus $E = 6.89 \times 10^{10}$Pa (10^7 psi) and material den-

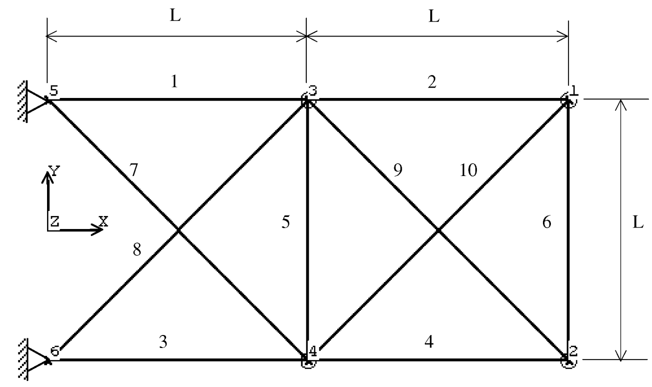

Fig. 5 Ten-bar planar truss structure: $L = 9.144$ m (360 in.).

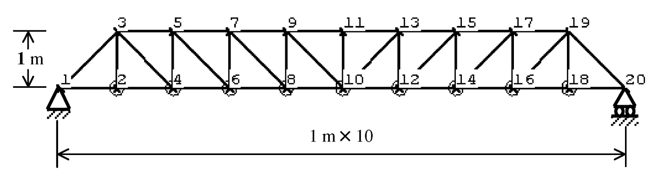

Fig. 6 Initial configuration design of a simply supported bridge.

sity $\rho = 2770$ kg/m^3 (0.1 lb$_m$/in.^{3}). The lower bound of the section area is set to be 0.645 cm^2 (0.1 in.^{2}) for all elements. A nonstructural mass of 454 kg (2.588 lb · s^2/in.) is attached at each of the four free nodes. For the initial design, all of the section areas are 20 cm^2.

The optimal solutions of the sectional designs, structural weight, and frequencies obtained by the present approach are listed in Tables 3 and 4, respectively, for comparison with Refs. 4 and 7. It is shown that the optimality criterion method can obtain excellent results comparable to those obtained with the SQP technique, even though there exist slight differences in the optimal designs of the sectional areas.

C. Simply Supported Bridge

To demonstrate the flexibility of the approach, a simply supported bridge is optimized for its weight minimization with several cases of frequency constraints. Members on the lower chord are represented by beam elements with fixed rectangular cross sections $B = 8$ cm and $H = 5$ cm. Others are modeled as bar elements with initial sectional areas $A = 1$ cm^2. Young's modulus is $E = 2.1 \times 10^{11}$Pa, and the material density is $\rho = 7800$ kg/m^3 for all elements. The initial configuration of the structure is shown in Fig. 6. A nonstructural mass $m = 10$ kg is attached at each of the nodes on the lower chord. In the optimization process, nodes on the upper chord can be shifted vertically, while nodes on the lower chord remain fixed. In addition, nodal coordinates and member areas are linked to maintain structural symmetry about the y–z plane. Thus, only 5 shape variables and 14 sizing variables will be redesigned for optimization. Lower bounds on the sections are 1 cm^2 for all bar elements.

First, the structure is optimized subject to a fundamental frequency constraint. Two different values, 20 and 25 Hz, are set as lower limits of the fundamental frequency. The optimal designs of the variables and structural weights (except nonstructural masses) are given in Table 5. Table 6 lists the first five natural frequencies at the initial and optimum design for comparison. It is seen from Table 5 that, except for shape modification, element areas only on the upper chord experience increments. The optimal configuration of the bridge for the constraint $\omega_1 \geq 25$ Hz is shown in Fig. 7.

Next, the structure is optimized with the second natural frequency constraint $\omega_2 \geq 40$ Hz. The optimal designs are also listed in Table 5 and 6, respectively. When the optimal design of the node y coordinates are compared, it is recognized that the structural shape is quite different from that under the fundamental frequency constraint. In addition, areas of elements 12 and 15 also experience increments.

Finally, the structure is optimized with multiple constraints of natural frequencies. The frequency constraints for the first case are $\omega_1 \geq 20$ and $\omega_2 \geq 40$ Hz and for the second case are $\omega_1 \geq 20$,

Table 3 Comparison of optimal designs of 10-bar planar truss on sections (square centimeters) under different frequency constraints

		Optimal design									
		$\omega_1 \geq 10$ Hz		$\omega_1 \geq 14$ Hz		$\omega_2 \geq 25$ Hz			$\omega_1 \geq 7, \omega_2 \geq 15, \omega_3 \geq 20$ Hz		
Element number	Initial design	Present method	Ref. 4	Present method	Ref. 7	Present method	Ref. 4	Ref. 7	Present method	Ref. 4	Ref. 7
1	20	90.340	90.074	220.680	219.903	48.932	49.659	48.123	32.456	36.584	38.245
2	20	24.172	28.619	48.043	47.916	34.984	46.595	35.832	16.577	24.658	9.916
3	20	90.340	90.074	220.680	219.903	48.932	49.646	48.200	32.456	36.584	38.619
4	20	24.172	28.619	48.043	47.916	34.984	46.588	35.884	16.577	24.658	18.232
5	20	0.645	0.645	0.645	0.645	15.041	14.158	14.826	2.115	4.167	4.419
6	20	0.645	0.645	0.645	0.645	7.789	8.914	7.632	4.467	2.070	4.194
7	20	49.220	48.885	124.095	123.626	41.226	40.061	41.103	22.810	27.032	20.097
8	20	49.220	48.885	124.095	123.626	41.226	40.061	41.181	22.810	27.032	24.097
9	20	27.433	32.308	54.847	54.677	13.449	25.910	13.200	17.490	10.346	13.890
10	20	27.433	32.308	54.847	54.677	13.449	25.819	13.187	17.490	10.346	11.452
Weight, kg	590.5	1132.5	1186.8	2646.5	2637.9	874.6	1018.7	871.9	553.8	594.0	537.0

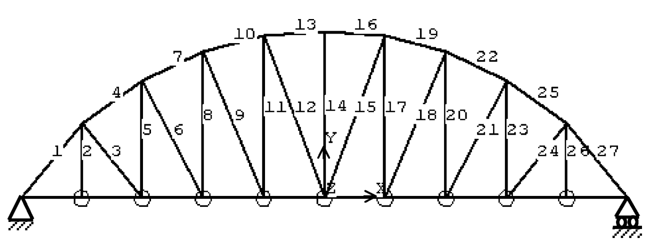

Fig. 7 Optimal configuration of the simply supported bridge with constraint $\omega_1 \geq 25$ Hz.

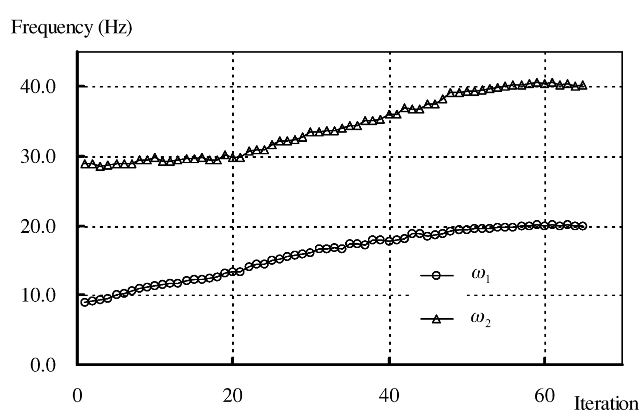

Fig. 8 Evolutionary histories of the frequencies with constraints of $\omega_1 \geq 20$ and $\omega_2 \geq 40$ Hz.

$\omega_2 \geq 40$, and $\omega_3 \geq 60$ Hz. The optimal solutions are listed in Table 5 and 6, respectively. Figure 8 shows the evolutionary histories of the constrained frequencies in the first case of frequency constraints.

D. Dome Structure

The dome structure[14] is studied for shape and sizing optimization with the fundamental frequency constraint. There are 52 elements linked into 8 groups, and all nodal coordinates are linked to keep the dome symmetric, as shown in Fig. 9. Shape variables are chosen as Z_1, X_2, Z_2, X_6, and Z_6 (in meters), respectively. Suppose Young's modulus to be $E = 2.1 \times 10^{11}$Pa, and the material density $\rho = 7850$ kg/m^3. The first frequency is required to exceed 32 Hz. Note that the fundamental frequency is a double repeated one and that the repeated frequencies always exist during the solution process because of the structural symmetry. The initial design of all element areas is 5 cm^2, and the lower bound is assumed to be 4 cm^2. Owing to the lack of nonstructural masses, the initial frequencies of the dome remain unchanged if all of the sections are modified uniformly on a fixed shape.[15]

Table 4 Optimal designs of the natural frequencies (hertz) under different frequency constraints

Frequency number	Initial design	$\omega_1 \geq 10$ Hz	$\omega_1 \geq 14$ Hz	$\omega_2 \geq 25$ Hz	$\omega_1 \geq 7$ $\omega_2 \geq 15$ $\omega_3 \geq 20$ Hz
1	6.02	10.00	14.00	8.05	7.01
2	18.15	13.90	18.01	25.00	17.30
3	19.39	22.23	29.40	25.00	20.00
4	34.07	25.85	34.55	26.91	20.10
5	39.05	37.32	49.36	32.90	30.86
6	44.46	39.99	53.11	41.11	32.67
7	45.92	65.62	85.10	62.17	48.28
8	50.63	69.12	90.40	64.44	52.31

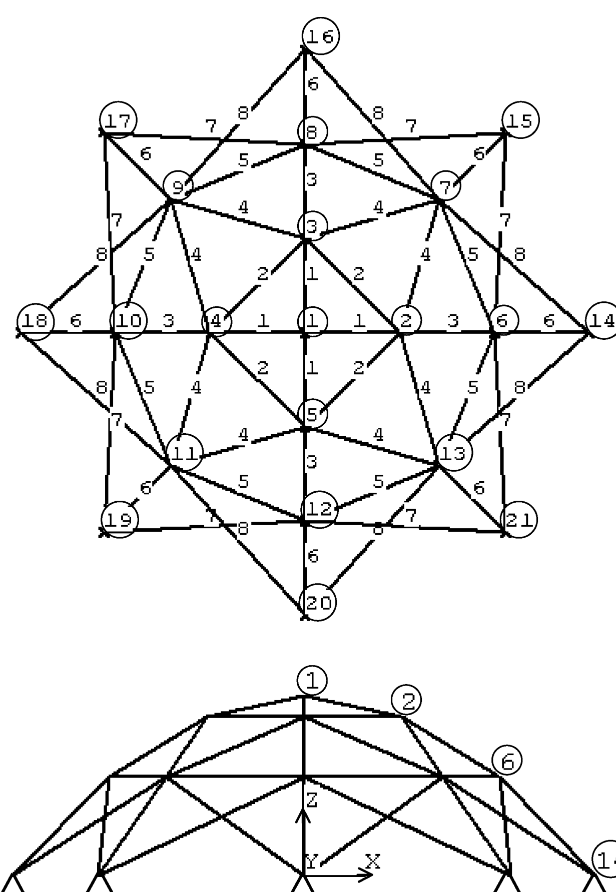

Fig. 9 Initial configuration design of the dome structure.

Table 5 Designs of node y coordinates (meters) and cross sections (square centimeters) at the initial and optimum under different frequency constraints

		Optimal designs				
Variable	Initial design	$\omega_1 \geq 20$ Hz	$\omega_1 \geq 25$ Hz	$\omega_2 \geq 40$ Hz	$\omega_1 \geq 20$ $\omega_2 \geq 40$ Hz	$\omega_1 \geq 20$ $\omega_2 \geq 40$ $\omega_3 \geq 60$ Hz
Y_3, Y_{19}	1.0	1.0114	1.2302	1.0792	1.2055	1.2086
Y_5, Y_{17}	1.0	1.5872	1.9345	1.4406	1.7109	1.5788
Y_7, Y_{15}	1.0	1.9746	2.4216	1.3788	1.8746	1.6719
Y_9, Y_{13}	1.0	2.1956	2.6781	0.9819	1.9050	1.7703
Y_{11}	1.0	2.2617	2.7411	0.7260	1.9735	1.8502
A_1, A_{27}	1.0	1.6619	2.2936	1.6113	2.5907	3.2508
A_2, A_{26}	1.0	1.0000	1.0000	1.0000	1.0000	1.2364
A_3, A_{24}	1.0	1.0000	1.0000	1.0000	1.0000	1.0000
A_4, A_{25}	1.0	1.6664	2.1995	1.3941	2.1926	2.5386
A_5, A_{23}	1.0	1.0000	1.0000	1.0000	1.0000	1.3714
A_6, A_{21}	1.0	1.0000	1.0000	1.0000	1.0000	1.3681
A_7, A_{22}	1.0	1.7267	2.1792	1.3616	2.1279	2.4290
A_8, A_{20}	1.0	1.0000	1.0000	1.0000	1.1199	1.6522
A_9, A_{18}	1.0	1.0000	1.0000	1.0000	1.1804	1.8257
A_{10}, A_{19}	1.0	1.7267	2.1199	1.2751	2.1247	2.3022
A_{11}, A_{17}	1.0	1.0000	1.0000	1.0000	1.4745	1.3103
A_{12}, A_{15}	1.0	1.0000	1.0000	1.6807	1.6531	1.4067
A_{13}, A_{16}	1.0	1.6902	2.1113	1.0000	2.0433	2.1896
A_{14}	1.0	1.0000	1.0000	1.0000	1.0000	1.0000
Weight, kg	336.3	351.7	361.8	343.4	360.8	366.5

Table 6 First five natural frequencies (hertz) under different frequency constraints

Frequency number	Initial design	$\omega_1 \geq 20$ Hz	$\omega_1 \geq 25$ Hz	$\omega_2 \geq 40$ Hz	$\omega_1 \geq 20$ $\omega_2 \geq 40$ Hz	$\omega_1 \geq 20$ $\omega_2 \geq 40$ $\omega_3 \geq 60$ Hz
1	8.89	20.00	25.00	7.54	20.01	20.01
2	28.82	31.32	30.36	40.00	40.29	41.84
3	46.92	47.17	44.80	53.20	52.43	62.22
4	63.62	62.24	59.57	63.76	66.67	73.17
5	76.87	76.78	73.39	81.97	81.54	88.43

Table 7 Initial and optimal designs of dome structure with a fundamental frequency constraint

		Optimal design with $\omega_1 \geq 32$ Hz	
Design variable	Initial design	With shape and sizing	With pure sizing
Z_1	9.25	10.166	——
X_2	5.00	6.407	——
Z_2	8.22	8.013	——
X_6	10.00	10.503	——
Z_6	5.14	4.893	——
A_1	5.00	4.000	4.000
A_2	5.00	4.000	4.000
A_3	5.00	4.204	4.000
A_4	5.00	4.000	4.000
A_5	5.00	4.000	5.510
A_6	5.00	4.000	4.000
A_7	5.00	7.018	11.092
A_8	5.00	9.377	8.731
Weight, kg	1740.6	2042.1	2341.3

Table 8 First five natural frequencies (hertz) with a fundamental frequency constraint

		Optimal design with $\omega_1 \geq 32$ Hz	
Frequency number	Initial design	With shape and sizing	With pure sizing
1	28.44	32.00	32.00
2	28.44	32.00	32.00
3	30.44	32.07	32.06
4	30.86	32.23	32.07
5	30.86	32.29	32.07

Table 7 presents the initial and optimal design of the variables. In addition, the first five frequencies are listed in Table 8. It is seen that these frequencies become very closely spaced after optimization.

In contrast, the dome is optimized again for pure sizing optimization with the fixed geometry of the initial design. The optimal solutions are also listed in Tables 7 and 8, respectively, for comparison. The final structural weight is greater than the former by 14.65%, and the first five natural frequencies become much closer to each other. This example demonstrates again that truss optimization on both shape and sizing variables is more efficient than that on sizing variables only. In other words, more effective design can be achieved by optimizing both shape and sizing variables.

VIII. Conclusions

In this paper, the sensitivity number of a design variable is defined for truss shape and sizing variables irrespective of their distinct natures. Based on the sensitivity numbers, these two types of variables are optimized in a unified design space for structural weight minimization. An optimality criteria algorithm in conjunction with the weighted sum of the sensitivity numbers is applied to optimization problems with multiple constraints of natural frequencies. The optimization process is implemented in such a way that the most efficient variables are identified and modified in priority so that the structural weight is indirectly minimized. Four typical trusses are employed to demonstrate the validity of the method. Moreover, structures with repeated frequencies can also be treated effectively with the proposed method.

At times, a frequency may be subject to a frequency-prohibited band or two-sided inequality constraint in practice.[1] In this case, the optimization process can be implemented twice with different one-sided inequality constraints to compare their final solutions. Only the final results can determine the optimal design.

Appendix: Theoretical Analysis of First Example

In Fig. 1, the area of the bars and position of supports are required to be optimally designed with a fundamental frequency constraint. Let us introduce an intermediate variable β, the angle between the horizontal line and the bar, to simplify the subsequent expression. From Fig. 1, we have the following geometric relations:

$$L = D/\cos\beta, \qquad y_s = D\tan\beta \tag{A1}$$

where L is the length of the bars and D the horizontal distance of the joint departing from the vertical support line, which remains fixed during the design process. Here y_s denotes the support position. The weight of the structure is computed as

$$W = 2\rho AL \tag{A2}$$

where ρ is the mass density and A the cross-sectional area of the bars.

The reduced global stiffness and mass matrices are, respectively,

$$[K] = \frac{2AE}{D}\begin{bmatrix}\cos^3\beta & 0\\ 0 & \sin^2\beta\cos\beta\end{bmatrix} \tag{A3}$$

$$[M] = \begin{bmatrix}\dfrac{2AD\rho}{3\cos\beta}+m & 0\\ 0 & \dfrac{2AD\rho}{3\cos\beta}+m\end{bmatrix} \tag{A4}$$

Hence, the two natural frequencies are obtained:

$$\omega_x^2 = \frac{2AE\cos^4\beta}{D(2AD\rho/3 + m\cos\beta)} \tag{A5a}$$

$$\omega_y^2 = \frac{2AE\sin^2\beta\cos^2\beta}{D(2AD\rho/3 + m\cos\beta)} \tag{A5b}$$

The ratio of the frequency in y axis to that in x axis, r, is

$$r = \omega_y/\omega_x = \tan\beta \tag{A6}$$

Equation (A6) implies that the frequency ratio of the system is independent of the cross-sectional area of the bars. From the basic theorem of dynamic analysis, one can conclude the following:

1) When $\beta < 45$ deg, the first frequency is along the y axis,

$$\omega_1^2 = \frac{2AE\sin^2\beta\cos^2\beta}{D(2AD\rho/3 + m\cos\beta)} \qquad (\beta < 45) \tag{A7}$$

2) When $\beta > 45$ deg, the first frequency is along the x axis,

$$\omega_1^2 = \frac{2AE\cos^4\beta}{D(2AD\rho/3 + m\cos\beta)} \qquad (\beta > 45) \tag{A8}$$

3) When $\beta = 45$ deg, an exact double repeated frequency emerges, and the fundamental frequency reaches its maximum:

$$\omega_1^2\big|_{\max} = \omega_x^2 = \omega_y^2 = \frac{AE}{D(4AD\rho/3 + \sqrt{2}m)} \tag{A9}$$

Subsequently, the design sensitivity analysis is performed. For simplicity of derivation but without loss of generality, we will treat the case of $0 < \beta \le 45$ deg, that is, $0 < y_s \le 1$, where $\omega_1 = \omega_y$ and $\omega_2 = \omega_x$.

The sensitivities of the first frequency and structural weight with respect to the bars' area can be computed as follows. From Eq. (A7), one obtains

$$\frac{\partial\omega_1^2}{\partial A} = \frac{2Em\sin^2\beta\cos^3\beta}{D(2AD\rho/3 + m\cos\beta)^2} \tag{A10}$$

It is obvious that the frequency sensitivity of the area is positive when $0 < \beta < 45$ deg, which means that increasing the area will certainly raise the fundamental frequency. From Eq. (A2), one obtains

$$\frac{\partial W}{\partial A} = \frac{2\rho D}{\cos\beta} \tag{A11}$$

Thus, the sensitivity number or the efficiency of the bars' area is calculated as

$$\alpha_{1A} = \frac{\partial\omega_1^2}{\partial W} = \frac{Em\sin^2\beta\cos^4\beta}{D^2\rho(2AD\rho/3 + m\cos\beta)^2} \tag{A12}$$

The positive sensitivity number of the frequency means that increasing the structural weight by means of the bars' area will raise the fundamental frequency. The sensitivity number or the efficiency of the position of the supports is then calculated by using the chain rule of differentiation. From Eq. (A1),

$$\frac{dy_s}{d\beta} = \frac{D}{\cos^2\beta} \tag{A13}$$

$$\frac{\partial W}{\partial y_s} = \frac{2\rho AD\sin\beta}{\cos^2\beta}\cdot\frac{\cos^2\beta}{D} = 2\rho A\sin\beta \tag{A14}$$

$$\frac{\partial\omega_1^2}{\partial y_s} = \frac{\partial\omega_1^2}{\partial\beta}\cdot\frac{d\beta}{dy_s} = \frac{2AEm\cos^4\beta\sin^3\beta}{D^2(2AD\rho/3 + m\cos\beta)^2} + \frac{4AE(\sin\beta\cos^5\beta - \cos^3\beta\sin^3\beta)}{D^2(2AD\rho/3 + m\cos\beta)} \tag{A15}$$

Therefore,

$$\alpha_{1y_s} = \frac{\partial\omega_1^2}{\partial W} = \frac{Em\sin^2\beta\cos^4\beta}{D^2\rho(2AD\rho/3 + m\cos\beta)^2} + \frac{2E(\cos^5\beta - \cos^3\beta\sin^2\beta)}{D^2\rho(2AD\rho/3 + m\cos\beta)} \tag{A16}$$

When Eq. (A12) is compared with Eq. (A16), it is not hard to find that the sensitivity number of the bars' area is only a part of that of the supports' position. That is, the latter is larger than the former by the second item in Eq. (A16), which is always positive in the region of $0 < \beta < 45$ deg or $0 < y_s < 1$.

Figure A1 shows the variations of the sensitivity numbers of the fundamental frequency for the two design variables (with $A = 0.2$ cm^2 and $D = 1.0$ m).

The results in the first example can be solved analytically. For a prespecified fundamental frequency limit ω_1^*, the constraint on the frequency has the form

$$\frac{2AE\sin^2\beta\cos^2\beta}{D(2AD\rho/3 + m\cos\beta)} = \left(\omega_1^*\right)^2 \tag{A17}$$

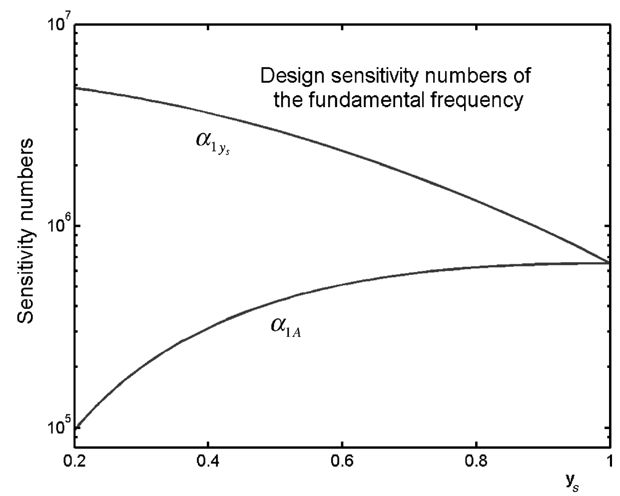

Fig. A1 Variations of the design sensitivity numbers with respect to the position of the supports.

Then, the area of the elements is calculated:

$$A = \frac{6mD\left(\omega_1^*\right)^2 \cos\beta}{3E\sin^2 2\beta - 4\rho D^2\left(\omega_1^*\right)^2} \qquad (A \geq \underline{A}) \tag{A18}$$

where $\underline{A}$ is the lower bound of the area. When Eqs. (A18) and (A1) are inserted into Eq. (A2), the structural weight becomes a function of only variable β as

$$W = 2\rho \times \frac{D}{\cos\beta} \times \frac{6mD(\omega_1^*)^2\cos\beta}{3E\sin^2 2\beta - 4\rho D^2\left(\omega_1^*\right)^2}$$

$$= \frac{12m\rho D^2\left(\omega_1^*\right)^2}{3E\sin^2 2\beta - 4\rho D^2\left(\omega_1^*\right)^2} \tag{A19}$$

The optimality condition for the weight is

$$\frac{\partial W}{\partial\beta} = \frac{12m\rho D^2\left(\omega_1^*\right)^2 \times 6E\sin 4\beta}{\left[3E\sin^2 2\beta - 4\rho D^2\left(\omega_1^*\right)^2\right]^2} = 0 \tag{A20}$$

which gives

$$\sin 4\beta = 0 \tag{A21}$$

Therefore, the optimal shape solution of the two-bar truss is achieved when

$$\beta = 45 \text{ deg} \tag{A22}$$

From which we can obtain

$$y_s = D \tag{A23}$$

This result means that the optimal shape of the two-bar truss always occurs at the state with the double repeated frequency. The structural weight at the optimum is found as

$$W^{\text{opt}} = \frac{12m\rho D^2\left(\omega_1^*\right)^2}{3E - 4\rho D^2\left(\omega_1^*\right)^2} \tag{A24}$$

For example, let $\omega_1^* = 628.32$ (rad/s), that is, 100 Hz, and $D = 1.0$ m; the optimal weight (in kilograms) is

$$W^{\text{opt}} = 0.598$$

and the optimal area (in square meters) of the elements is achieved by Eq. (A18):

$$A^{\text{opt}} = 0.271 \times 10^{-4}$$

which is identical to the numberical results in the first example.

On the other hand, using Eqs. (A1), (A2), and (A7), one can easily solve the dual problem of the foregoing such as maximizing the fundamental frequency for a prescribed weight W_0. The same solution $y_s = D$ can also be obtained for the problem. In fact, the constraint on the weight will have the form

$$W_0 = 2\rho AL \tag{A25}$$

or

$$A = W_0\cos\beta/2\rho D \qquad (A \geq \underline{A}) \tag{A26}$$

When Eq. (A26) is substituted back into Eq. (A7), the frequency will be

$$\omega_1^2 = \frac{W_0 E\sin^2\beta\cos^2\beta}{\rho D^2[(W_0/3) + m]} \tag{A27}$$

Let the derivative of the frequency with respect to β be equal to zero,

$$\frac{\partial\omega_1^2}{\partial\beta} = \frac{W_0 E\sin 4\beta}{2\rho D^2[(W_0/3) + m]} = 0 \tag{A28}$$

which would give the same equation as Eq. (A21). For example, assume $W_0 = 0.598$ kg, the maximal fundamental frequency [in radians per second (or hertz)] is obtained from Eq. (A27):

$$\omega_1^{\max} = 628.32 \quad \text{or} \quad (100)$$

The area in square meters of the elements is

$$A^{\text{opt}} = \frac{0.598}{7800\sqrt{8}} = 0.271 \times 10^{-4}$$

which is again identical to the optimal results in the first example.

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